

A simple example of temporal cloaking

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Abstract

Recently the temporal cloaking was widely studied in the physical literature using the method of transformation optics (cf. M. McCall et al [9] and others). We give a simple example of temporal cloaking based on the change of coordinates and the reduction to the initial-boundary value problem with additional interior conditions.

Introduction and the main result

The transformation optics approach, combined with the use of metamaterials, leads to a remarkable progress in the construction of spatial cloaking devices and other problems (cf. J.B. Pendry et al [10], U. Leonhardt [8] and others). The mathematical analysis of the spatial cloaking was done by A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann [2], [3], [4], [5] (see the survey article [4] for additional references).

The temporal cloaking uses the transformation of variables in spacetime and it is conceptually simpler than the spatial cloaking. The important works on the temporal cloaking were done by McCall et al [9], Kinster et al [7], M. Fridman [1] and others.

Our approach to the temporal cloaking is different and uses the reduction to the initial-boundary value problem outside the cloaked domain with some additional conditions.

We shall describe our results in details.

Let D be a bounded domain in \mathbb{R}^n . Consider the initial-boundary value problem:

$$\begin{aligned} (1) \quad & Lu(x_0, x) \stackrel{def}{=} \frac{\partial^2 u(x_0, x)}{\partial x_0^2} - \sum_{k=1}^n \frac{\partial^2 u(x_0, x)}{\partial x_k^2} = 0, \quad (x_0, x) \in \mathbb{R} \times D, \\ (2) \quad & u = 0 \quad \text{for} \quad x_0 \ll 0, \quad x \in D, \\ (3) \quad & u|_{\mathbb{R} \times D} = f, \end{aligned}$$

where $x = (x_1, \dots, x_n)$, f has a compact support in $\mathbb{R} \times \partial D$.

Make a change of variables in $\mathbb{R} \times D$

$$(4) \quad y_0 = \varphi_0(x_0, x), \quad y = x.$$

Let $\tilde{L}v(y_0, y) = 0$ be the equation (1) in (y_0, y) -coordinates, where $v(y_0, y) = u(x_0, x)$, (y_0, y) and (x_0, x) are related by (4). In particular case when

$$\varphi_0(x_0, x) = x_0 - c(x),$$

the equation $\tilde{L}v(y_0, y) = 0$ has the following form

$$\begin{aligned} (5) \quad & \left(1 - \sum_{j=1}^n c_{y_j}^2(y)\right) \frac{\partial^2 v(y_0, y)}{\partial y_0^2} - \sum_{j=1}^n \frac{\partial}{\partial y_j} \left(c_{y_j} \frac{\partial v}{\partial y_0} \right) \\ & - \sum_{j=1}^n c_{y_j} \frac{\partial^2 v}{\partial y_j \partial y_0} - \sum_{j=1}^n \frac{\partial^2 v(y_0, y)}{\partial y_j^2} = 0. \end{aligned}$$

We assume, for the simplicity, that

$$(6) \quad \sum_{j=1}^n c_{y_j}^2(y) < 1.$$

However, this restriction is not necessary (see Remark 1).

We specify that

$$\begin{aligned} (7) \quad & \varphi_0(x_0, x) = \varphi_0^+(x_0, x) = x_0 + c(x) \quad \text{for} \quad x_0 > 0, \\ (8) \quad & \varphi_0(x_0, x) = \varphi_0^-(x_0, x) = x_0 - c(x) \quad \text{for} \quad x_0 < 0, \end{aligned}$$

where $c(x) = c_0 \chi(x)$, $c_0 > 0$, $\chi(x) \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) = 0$ for $|x| > c_1$, $\chi(x) = 1$ for $|x| < \frac{c_1}{2}$, $0 < \chi(x) < 1$ for $\frac{c_1}{2} < |x| < c_1$, the ball $|x| \leq c_1$ is inside D .

Denote by Y^+ and Y^- the sets $\{y_0 > c(y), y \in \mathbb{R}^n\}$ and $\{y_0 < -c(y), y \in \mathbb{R}^n\}$, respectively.

Let

$$(9) \quad Y_0 = \{(y_0, y) : -c(y) < y_0 < c(y), |y| < c_1\}.$$

Note that

$$(10) \quad \overline{Y}^+ \cup \overline{Y}^- = \mathbb{R}^{n+1} \setminus Y_0.$$

Consider the equation

$$(11) \quad \tilde{L}^- v^-(y_0, y) = 0 \quad \text{in} \quad Y^- \cap (\mathbb{R} \times D)$$

corresponding to the change of variable $y_0 = \varphi_0^-(x) = x_0 + c(x)$, $x_1 < 0$, with the initial condition

$$(12) \quad v^- = 0 \quad \text{for} \quad y_0 \ll 0, y \in D,$$

and the boundary condition

$$(13) \quad v^- = f \quad \text{on} \quad (\mathbb{R} \times \partial D) \cap Y^-.$$

This problem is well-posed and has a unique smooth solution $v^-(y_0, y)$. Now consider the initial-boundary value problem for

$$(14) \quad \tilde{L}^+ v^+(y_0, y) = 0 \quad \text{in} \quad Y^+ \cap (\mathbb{R} \times D),$$

corresponding to the change of variable $y_0 = \varphi_0^+(x) = x_0 - c(x)$, with the boundary condition

$$(15) \quad v^+ = f \quad \text{on} \quad (\mathbb{R} \times \partial D) \cap Y^+,$$

and the initial conditions

$$(16) \quad v^+ \Big|_{\partial Y^+ \cap D} = v^- \Big|_{\partial Y^- \cap D}, \quad \frac{\partial v^+}{\partial y_0} \Big|_{\partial Y^+ \cap D} = \frac{\partial v^-}{\partial x_0} \Big|_{\partial Y^- \cap D}.$$

Note that we assume that $v^- \Big|_{\partial Y^- \cap D}$ and $\frac{\partial v}{\partial y_0} \Big|_{\partial Y^- \cap D}$ are already known.

This initial-boundary value problem also has a unique smooth solution. Therefore we can determine a function $v(y_0, y)$ such that $v = v^+$ in $Y^+ \cap$

$(\mathbb{R} \times D)$, $v = v^-$ in $Y^- \cap (\mathbb{R} \times D)$, $\tilde{L}^+ v^+ = 0$ in $Y^+ \cap \mathbb{R} \times D$, $\tilde{L}^- v^- = 0$ in $Y^- \cap (\mathbb{R} \times D)$, $v = v^- = 0$ for $y_0 \ll 0$, $y \in D$, $v|_{\mathbb{R} \times \partial D} = f$ and v^+ and v^- satisfy the conditions (16).

Once $v(y_0, y)$ is given we determine $u(x_0, x)$ such that $u(x_0, x) = u^+(x_0, x)$ for $x_0 > 0$, $u^+(x_0, x) = v^+(y_0, y)$, where $x_0 = y_0 - c(y)$, $x = y$, $(y_0, y) \in Y^+$. Analogously, $u = u^-(x_0, x) = v^-(y_0, y)$, where $x_0 = y_0 + c(y)$, $x = y$, $(y_0, y) \in Y^-$.

Then $u^+(x_0, x)$ and $u^-(x_0, x)$ satisfy the wave equation (1) for $x_0 > 0$ and $x_0 < 0$, respectively. Since the conditions (16) are satisfied, we have that

$$(17) \quad \begin{aligned} \lim_{\substack{x \rightarrow 0 \\ x_0 < 0}} u^-(x_0, x) &= \lim_{\substack{x \rightarrow 0 \\ x_0 > 0}} u^+(x_0, x) \\ \lim_{\substack{x \rightarrow 0 \\ x_0 < 0}} \frac{\partial u^-(x_0, x)}{\partial x_0} &= \lim_{\substack{x \rightarrow 0 \\ x_0 > 0}} \frac{\partial u^+(x_0, x)}{\partial x_0} \end{aligned}$$

Therefore $u(x_0, x) = u^+(x_0, x)$ for $x_0 > 0$ and $u(x_0, x) = u^-(x_0, x)$ for $x_0 < 0$ satisfied (1) in $\mathbb{R} \times D$ and also satisfies the initial and boundary conditions (2), (3).

Since $c(x) = 0$ in $D \setminus \overline{B}$, where $B = \{x : |x| < c_1\}$ we have that $x_0 = y_0$, $x = y$ in $\mathbb{R} \times (D \setminus \overline{B})$. Thus

$$(18) \quad u(x_0, x) = v(x_0, x) \quad \text{in } \mathbb{R} \times (D \setminus \overline{B}).$$

Therefore the boundary data of $u(x_0, x)$ and $v(y_0, y)$ on $\mathbb{R} \times \partial D$ are the same. This implies that one can not distinguish between $u(x_0, x)$ in $\mathbb{R} \times D$ and $v(y_0, y)$ in $(\mathbb{R} \times D) \cap (Y^+ \cup Y^-)$. Since the domain Y_0 is outside of $\overline{Y}^+ \cup \overline{Y}^-$, Y_0 is a cloaking domain and the observer on $\mathbb{R} \times \partial D$ does not suspect its existence.

Summarizing the results of this section we get the following theorem:

Theorem 1. *Let domains Y_0, Y^+, Y^- be the same as in (9), (10) and let operators \tilde{L}^- and \tilde{L}^+ be the same as in (11) and (14). Consider the solution $v^-(y_0, y)$ of the initial-boundary value problem (11), (12), (13) in $Y^- \cap (\mathbb{R} \times D)$.*

Let $v^+(y_0, y)$ be the solution in $Y^+ \cap (\mathbb{R} \times D)$ of the initial-boundary value problem (14), (15), (16), where $v^-(y_0, y)$ is the same as in (11), (12), (13). Denote $v(y_0, y) = v^+(y_0, y)$ in $Y^+ \cap (\mathbb{R} \times D)$, $v = v^-(y_0, y)$ in $Y^- \cap (\mathbb{R} \times D)$. Since $v^- = v^+$ in $\overline{Y}^+ \cap \overline{Y}^-$ we have that $v(y_0, y)$ is defined and smooth in $(\mathbb{R} \times D) \setminus Y_0$.

Let $u^+(x_0, x) = v^+(y_0, y)$ for $x_0 > 0$, where $x = y, x_0 = y_0 - c(y)$ and let $u^-(x_0, x) = v^-(y_0, y)$ for $x_0 < 0$ where $x = y, x_0 = y_0 + c(y)$. It follows from (16) that $u(x_0, x) = u^+(x_0, x)$ for $x_0 > 0, u(x_0, x) = u^-(x_0, x)$ for $x_0 < 0$ extends to a smooth function in $\mathbb{R} \times D$ that satisfies (1), (2), (3). The boundary measurements of $v(y_0, y)$ defined in $(\mathbb{R} \times D) \setminus Y_0$ and $u(x_0, x)$ defined in $\mathbb{R} \times D$ are equal.

Thus Y_0 is a perfect cloak.

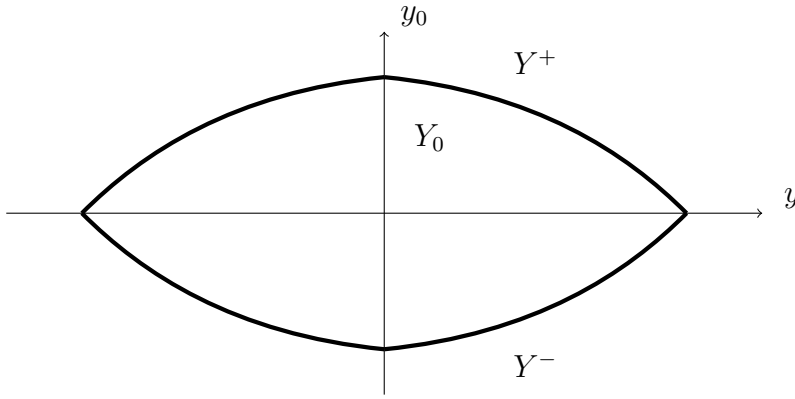


Fig. 1. Domain Y_0 is a cloaked region.

Remark 1. If the condition (6) is not satisfied then the plane $y_0 = \text{const}$ is not space-like for \tilde{L}^\pm . However the surfaces $y_0 \mp c(y) = \text{const}$ are space-like for \tilde{L}^\pm . Therefore the initial boundary value problem for \tilde{L}^\pm with the initial conditions on $y_0 \mp c(y) = 0$ is perfectly well-posed (see [H]).

Remark 2. Note that the domains Y_0, Y^+, Y^- and the changes of variables $x_0 = y_0 \pm c(y), x = y$, do not depend on the form of operator (1). Therefore Theorem 1 applies to any hyperbolic initial-boundary value problem.

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